

PIR Array Codes with Optimal PIR Rates

Simon R. Blackburn

Dept. of Mathematics

Royal Holloway University of London
Egham, Surrey TW20 0EX, United Kingdom
Email: s.blackburn@rhul.ac.uk

Tuvi Etzion

Dept. of Computer Science

Technion-Israel Institute of Technology
Haifa 32000, Israel
Email: etzion@cs.technion.ac.il

Abstract—There has been much recent interest in Private Information Retrieval (PIR) in models where a database is stored across several servers using coding techniques from distributed storage, rather than being simply replicated. In particular, a recent breakthrough result of Fazeli, Vardy and Yaakobi introduces the notion of a PIR code and a PIR array code, and uses this notion to produce efficient protocols.

In this paper we are interested in designing PIR array codes. We consider the case when we have m servers, with each server storing a fraction $(1/\omega)$ of the bits of the database; here ω is a fixed rational number with $\omega > 1$. We study the maximum PIR rate of a PIR array code with the k -PIR property (which enables a k -server PIR protocol to be emulated on the m servers), where the PIR rate is defined to be k/m . We present upper bounds on the achievable rate, some constructions, and ideas how to obtain PIR array codes with the highest possible PIR rate. In particular, we present constructions that asymptotically meet our upper bounds, and the exact largest PIR rate is obtained when $1 < \omega \leq 2$.

I. INTRODUCTION

A Private Information Retrieval (PIR) protocol allows a user to retrieve a data item from a database, in such a way that the servers storing the data will get no information about which data item was retrieved. The problem was introduced in [4]. The protocol to achieve this goal assumes that the servers are curious but honest, so they don't collude. It is also assumed that the database is error-free and synchronized all the time. For a set of k servers, the goal is to design a k -server PIR protocol, in which the efficiency of the PIR is measured by the total number of bits transmitted by all parties involved. This model is called a *information-theoretic* PIR; there is also *computational* PIR, in which the privacy is defined in terms of the inability of a server to compute which file was retrieved in reasonable time [9]. In this paper we will be concerned only with information-theoretic PIR.

The classical model of Private Information Retrieval assumes that each server stores a copy of an n -bit database, so the *storage overhead*, namely the total number of bits stored by all servers, is nk . However, recent work combines PIR protocols with techniques from distributed storage (where each server stores only some of the database) to reduce the storage overhead. This approach was first considered in [10], and several papers developed this direction further [1], [3], [5], [6], [7], [11], [12], [13]. Our discussion will follow the breakthrough

approach presented by Fazeli, Vardy, and Yaakobi [6], [7], which shows that m servers (for some $m > k$) may emulate a k -server PIR protocol with storage overhead significantly lower than nk .

Fazeli et al [7] introduce the key notion of a $[t \times m, s]$ k -PIR array code, which is defined as follows. Let x_1, x_2, \dots, x_s be a basis of a vector space of dimension p (over some finite field \mathbb{F}). A $[t \times m, s]$ array code is simply a $t \times m$ array, each entry containing a linear combination of basis elements x_i . A $[t \times m, s]$ array code satisfies the k -PIR property (or is a $[t \times m, s]$ k -PIR array code) if for every $i \in \{1, 2, \dots, s\}$ there exist k pairwise disjoint subsets S_1, S_2, \dots, S_k of columns so that for all $j \in \{1, 2, \dots, k\}$ the element x_i is contained in the linear span of the entries of the columns S_j . The following example of a (binary) $[7 \times 4, 12]$ 3-PIR array code is taken from [7]:

x_1	x_2	x_3	$x_1 + x_2 + x_3$
x_2	x_3	x_1	x_6
x_4	x_5	$x_4 + x_5 + x_6$	x_4
x_5	x_6	x_8	x_9
x_7	$x_7 + x_8 + x_9$	x_9	x_7
x_8	x_{10}	x_{11}	x_{12}
$x_{10} + x_{11} + x_{12}$	x_{11}	x_{12}	x_{10}

The 3-PIR property means that for all $j \in \{1, 2, \dots, 12\}$ we can find 3 disjoint subsets of columns whose entries span a subspace containing x_j . For example, x_5 is in the span of the entries in the subsets $\{1\}$, $\{2\}$ and $\{3, 4\}$ of columns; x_{11} is in the span of the entries in the subsets $\{1, 4\}$, $\{2\}$ and $\{3\}$ of columns.

In the example above, many of the entries in the array consist of a single basis element; we call such entries *singletons*.

Fazeli et al use a $[t \times m, s]$ k -PIR array code as follows. The database is partitioned into s parts x_1, x_2, \dots, x_s , each part encoded as an element of the finite field \mathbb{F} . Each of a set of m servers stores t linear combinations of these parts; the j th server stores linear combinations corresponding to the j th column of the array code. We say that the j th server has t *cells*, and stores one linear combination in each cell. They show that the k -PIR property of the array code allows the servers to emulate all known efficient k -server PIR protocols. But the storage overhead is ntm/s , and this can be significantly smaller than nk if a good array code (namely one where tm/s is much smaller than k) is used. Define $\omega = s/t$, so

ω can be thought of as the reciprocal of the proportion of the database stored on each server. For small storage overhead, we would like the ratio

$$\frac{nk}{ntm/s} = \omega \frac{k}{m} \quad (1)$$

to be as large as possible. We define the *PIR rate* (rate in short) of a $[t \times m, s]$ k -PIR array code to be k/m (this rate should not be confused with the rate of the code). In applications, we would like the rate to be as large as possible for several reasons: when ω , which represents the amount of storage required at each server, is fixed such schemes give small storage overhead compared to nk (see (1)); we wish to use a minimal number m of servers, so m should be as small as possible; large values of k are desirable, as they lead to protocols with lower communication complexity. We will fix the number t of cells in a server, and the proportion $1/\omega$ of the database stored per server and we seek to maximise the PIR rate. Hence, we define $g(\omega, t)$ to be the largest rate of a $[t \times m, s]$ k -PIR array code when ω and t (and so p) are fixed. We define $g(\omega) = \lim_{t \rightarrow \infty} g(\omega, t)$.

Most of the analysis in [6], [7] was restricted to the case $t = 1$. The following two results presented in [7] are the most relevant for our discussion. The first result corresponds to the case where each server holds a single cell, i.e. we have a PIR code (not an array code with $t > 1$).

Theorem 1. *For any given positive integer ω , $g(\omega, 1) = (2^{\omega-1})/(2^\omega - 1)$.*

The second result is a consequence of the only construction of PIR array codes given in [7] which is not an immediate consequence of the constructions for PIR codes.

Theorem 2. *For any integer $\omega \geq 3$, we have $g(\omega, \omega - 1) \geq \omega/(2\omega - 1)$.*

The goal of this paper is first to generalize the results of Theorems 1 and 2 and to find codes with better rates for a given ω . We would like to find out the behavior of $g(\omega, t)$ as a function of t . This will be done by providing several new constructions for k -PIR array codes which will imply lower bounds on $g(\omega, t)$ for a large range of pairs (ω, t) . This will immediately imply a related bound on $g(\omega)$ for various values of ω . Contrary to the construction in [7], the value of ω in our constructions is not necessarily an integer (this possible feature was mentioned in [7]): each rational number greater than one will be considered. We will also provide various upper bounds on $g(\omega, t)$, and related upper bounds on $g(\omega)$. It will be proved that some of the upper bounds on $g(\omega, t)$ are tight and also our main upper bound on $g(\omega)$ is tight.

To summarise, our notation used in the remainder of the paper is given by:

- 1) n - the number of bits in the database.
- 2) s - number of parts the database is divided into.

The parts will be denoted by x_1, x_2, \dots, x_s .

- 3) $\frac{1}{\omega}$ - the fraction of the database stored on a server.
- 4) m - the number of servers (i.e. the number of columns in the array).
- 5) t - number of cells in a server (or the number of rows in the array); so $t = s/\omega$.
- 6) k - the array code allows the servers to emulate a k -PIR protocol.
- 7) $g(\omega, t)$ - the largest PIR rate of a $[t \times m, s]$ k -PIR array code.
- 8) $g(\omega) = \lim_{t \rightarrow \infty} g(\omega, t)$.

Clearly, a PIR array code is characterized by the parameters, ω , t , k , and m (the integer n does not have any effect on the other parameters, except for some possible divisibility conditions). In [7], where the case $t = 1$ was considered, the goal was to find the smallest m for given ω and k . We will also fix two parameters, t and ω , and we will examine the ratio k/m (which might require both k and m to be large). The search for the smallest value of m when t , ω and k are fixed is left for future work.

The rest of this paper is organized as follows. In Section II we present a simple upper bound on the value of $g(\omega)$. Though this bound is attained, we prove that $g(\omega, t) < g(\omega)$ for any fixed values of ω and t . We will also state a more complex upper bound on $g(\omega, t)$ for various pairs (ω, t) , and it will be shown to be attainable for $1 < \omega \leq 2$. In Section III we present a range of explicit constructions. In Subsection III-A we consider the case where $1 < \omega \leq 2$. In Subsection III-B we consider the case where ω is rational number greater than 2. In Section IV we present a construction in which at least $t - 1$ cells in each server have singletons. In Section IV we present a construction in which at least $t - 1$ cells in each server have singletons and its rate asymptotically meets the upper bound. We believe that this construction always produces the best bounds and prove this statement in some cases. For lack of space we omit some proofs and some constructions. These can be found in the full version of this paper [2].

II. UPPER BOUNDS ON THE PIR RATE

In this section we will be concerned first with a simple general upper bound (Theorem 3) on the rate of a k -PIR array code for a fixed value of ω with $\omega > 1$. This bound cannot be attained, but is asymptotically optimal (as $t \rightarrow \infty$). This will motivate us to give a stronger upper bound (Theorem 4) on the rate $g(\omega, t)$ of a $[t \times m, \omega t]$ k -PIR array code for various values of t that can sometimes be attained.

Theorem 3. *For each rational number $\omega > 1$ we have that $g(\omega) \leq (\omega + 1)/(2\omega)$. There is no t such that $g(\omega, t) = (\omega + 1)/(2\omega)$.*

Proof: Suppose we have a $[t \times m, p]$ k -PIR array code with $s/t = \omega$. To prove the theorem, it is sufficient to show that $k/m < (\omega + 1)/(2\omega)$. Recall that we are assuming, without loss of generality, that if x_i can be

derived from information on a certain server, then the singleton x_i is stored as the value of one of the cells of this server.

Let α_i be the number of servers which hold the singleton x_i in one of their cells. Since each server has t cells, we find that $\sum_{i=1}^s \alpha_i \leq tm$, and so the average value of the integers α_i is at most $tm/p = m/\omega$. So there exists $u \in \{1, 2, \dots, s\}$ such that $\alpha_u \leq m/\omega$ (and we can only have $\alpha_u = m/\omega$ when $\alpha_i = m/\omega$ for all $i \in \{1, 2, \dots, s\}$). Let $S^{(1)}, S^{(2)}, \dots, S^{(k)} \subseteq \{1, 2, \dots, m\}$ be disjoint sets of servers, chosen so the span of the cells in each subset of servers contains x_u . Such subsets exist, by the definition of a k -PIR array code. If no server in a subset $S^{(j)}$ contains the singleton x_u , the subset $S^{(j)}$ must contain at least two elements (because we are assuming, without loss of generality, that if x_i can be derived from information on a certain server, then the singleton x_i is stored as the value of one of the cells of this server.). So at most α_u of the subsets $S^{(j)}$ are of cardinality 1. In particular, this implies that $k \leq \alpha_u + (m - \alpha_u)/2$. Hence

$$\begin{aligned} \frac{k}{m} &\leq \frac{\alpha_u + (m - \alpha_u)/2}{m} = \frac{1}{2} + \frac{\alpha_u}{2m} \\ &\leq \frac{1}{2} + \frac{m/\omega}{2m} = \frac{1}{2} + \frac{1}{2\omega} = \frac{\omega + 1}{2\omega}. \end{aligned} \quad (2)$$

We can only have equality in (2) when $\alpha_i = m/s$ for all $i \in \{1, 2, \dots, s\}$, which implies that all cells in every server are singletons. But then the span of subset of servers contains x_i if and only if it contains server with a cell x_i , and so $k \leq \alpha_i = m/\omega$. But this implies that the rate k/m of the array code is at most $1/\omega = 2/(2\omega)$. This contradicts the assumption that the rate of the array code is $k/m = (\omega + 1)/(2\omega)$, since $\omega > 1$. So $k/m < (\omega + 1)/(2\omega)$, as required. ■

Theorem 4. For any integer $t \geq 2$ and any positive integer d , we have

$$g(1 + \frac{d}{t}, t) \leq \frac{(2d+1)t + d^2}{(t+d)(2d+1)} = 1 - \frac{d^2 + d}{(t+d)(2d+1)}.$$

Remark 1. Note that by considering a rational number $s > 1$ as $s = 1 + d/t$, where d is a positive integer, all admissible rational numbers are considered. Since $p = st$ must be an integer and t can be any positive integer, it follows that $s = p/t > 1$ can be always written as $s = 1 + d/t$, where d is a positive integer.

Corollary 1. For any positive integers δ and τ , such that $\gcd(\delta, \tau) = 1$ and for every integer $\ell \geq 1$ we have

$$g(1 + \frac{\delta}{\tau}, t) \leq \frac{\ell\delta^2 + \tau + 2\ell\tau}{2\ell\delta^2 + \delta + \tau + 2\ell\tau},$$

where $t = \ell\tau$.

III. CONSTRUCTIONS AND LOWER BOUNDS

In this section we will propose various constructions for PIR array codes; these yield lower bounds on $g(\omega, t)$

and on $g(\omega)$. The constructions yield an improvement on the lower bound on $g(\omega)$ implied by Theorem 2. They also cover all rational values of $\omega > 1$, and not just integer values of ω . We are interested in constructions in which the number of servers is as small as possible, although the main goal in this paper is providing a lower bound on the rate. In the constructions below, we use Hall's marriage Theorem [8]:

Theorem 5. In a finite bipartite graph $G = (V_1 \cup V_2, E)$, there is perfect matching if for each subset X of V_1 , the number of vertices in V_2 connected to vertices of X has at least size $|X|$.

Corollary 2. A finite regular bipartite graph has a perfect matching.

A. Constructions for $1 < \omega \leq 2$

In this subsection we present constructions for PIR array codes when ω is a rational number greater than 1 and smaller than or equal to 2. The first construction will be generalized in in Subsection III-B and Section IV, when ω is any rational number greater than 1, but the special case considered here deserves separate attention for three reasons: it is simpler than its generalization; the constructed PIR array code attains the bound of Theorem 4, while we do not have a proof of a similar result for the generalization; and finally the analysis of the generalization is slightly different.

Construction 1. ($\omega = 1 + d/t$ and $s = t + d$ for $t > 1$, d a positive integer, $1 \leq d \leq t$).

Let ϑ be the least common multiple of d and t . There are two types of servers. Servers of Type A store t singletons. Each possible t -subset of parts occurs ϑ/d times as the set of singleton cells of a server, so there are $\binom{p}{t} \vartheta/d$ servers of Type A. Each server of Type B has $t - 1$ singleton cells in $t - 1$ cells; the remaining cell stores the sum of the remaining $s - (t - 1) = d + 1$ parts. Each possible $(t - 1)$ -set of singletons occurs ϑ/t times, so there are $\binom{s}{t-1} \vartheta/t$ servers of Type B.

Theorem 6. For any given $t > 1$ and $1 \leq d \leq t$,

$$g(1 + d/t, t) \geq \frac{(2d+1)t + d^2}{(t+d)(2d+1)}.$$

Proof: The total number of servers in Construction 1 is $m = \binom{t+d}{t} \vartheta/d + \binom{t+d}{d+1} \vartheta/t$. We now calculate k such that Construction 1 has the k -PIR property. To do this, we compute for each i , $1 \leq i \leq s$, a collection of pairwise disjoint sets of servers, each of which can recover the part x_i .

There are $\binom{t+d-1}{t-1} \vartheta/d$ servers of Type A containing x_i as a singleton cell. Let V_1 be the set of $\binom{t+d-1}{t-1} \vartheta/d$ remaining servers of Type A. There are $\binom{t+d-1}{t-2} \vartheta/t$ servers of Type B containing x_i as a singleton cell. Let V_2 be the set of $\binom{t+d-1}{t-1} \vartheta/t$ remaining servers of Type B.

We define a bipartite graph $G = (V_1 \cup V_2, E)$ as follows. Let $v_1 \in V_1$ and $v_2 \in V_2$. Let $X_1 \subseteq$

$\{x_1, x_2, \dots, x_s\}$ be the set of t singleton cells of the server v_1 . Let $X_2 \subseteq \{x_1, x_2, \dots, x_s\}$ be the parts involved in the non-singleton cell of the server v_2 . (So X_2 is the set of $d+1$ parts that are not singleton cells of v_2 . Note that $x_i \in X_2$.) We draw an edge from v_1 to v_2 exactly when $X_2 \setminus \{x_i\} \subseteq X_1$. Note that v_1 and v_2 are joined by an edge if and only if the servers v_1 and v_2 can together recover x_i .

The degrees of the vertices in V_1 are all equal; the same is true for the vertices in V_2 . Moreover, $|V_1| = \binom{t+d-1}{t} \vartheta/d = \binom{t+d-1}{t-1} \vartheta/t = |V_2|$. So G is a regular graph, and hence by Corollary 2 there exists a perfect matching in G . The edges of this matching form $|V_1|$ disjoint sets of servers, each of which can recover x_i . Thus, we have that $k = \binom{t+d-1}{t-1} \vartheta/d + \binom{t+d-1}{t-2} \vartheta/t + \binom{t+d-1}{t} \vartheta/d = m - \binom{t+d-1}{t} \vartheta/d$.

Finally, some simple algebraic manipulation shows us that

$$g(1+d/t, t) \geq \frac{k}{m} = \frac{(2d+1)t + d^2}{(t+d)(2d+1)}.$$

Corollary 3.

(i) For any given t and d , $1 \leq d \leq t$, when $\omega = 1+d/t$ we have

$$g(\omega, t) = 1 - \frac{d^2 + d}{(t+d)(2d+1)} = \frac{\omega + 1 + 1/d}{(2 + 1/d)\omega}.$$

(ii) For any rational number $1 < \omega \leq 2$, we have $g(\omega) = (\omega + 1)/(2\omega)$.

(iii) $g(2, t) = (3t + 1)/(4t + 2)$.

Construction 2. ($\omega = 1+d/t$, $s = t+d$, and there exists a Steiner system $S(d, d+1, s)$)

Let \mathcal{S} be a $S(d, d+1, s)$ Steiner system on the set of points $\{1, 2, \dots, s\}$. We define servers of two types. There are $\binom{t+d}{t} = \binom{t+d}{d}$ servers of Type A: each server stores a different subset of parts in t singleton cells. There are $\frac{d}{d+1} \binom{t+d}{d}$ servers of Type B, indexed by a set that repeats each of the $\frac{1}{d+1} \binom{t+d}{d}$ blocks $B \in \mathcal{S}$ a total of d times. One cell in a server of Type B contains the sum $\sum_{i \in B} x_i$; the remaining $t-1$ cells contain the $t-1$ parts not involved in this sum.

The PIR rate of Construction 2 attains the upper bound of Theorem 4 using fewer servers than in Construction 1. Unfortunately, Construction 2 can be applied on a limited number of parameters since the number of possible Steiner systems of this type is limited, and the number of known ones is even smaller.

B. Constructions when $\omega > 2$ is rational

We do not know the exact value of the asymptotic rate $g(\omega, t)$ of PIR codes when $\omega > 2$. These values will be considered in this subsection. We present only the bounds implied by the constructions given in [2]. In all the constructions there exist servers with fewer than $t-1$ singletons.

Theorem 7. For any given integers $r > 1$ and $r \leq t$, and $1 \leq d \leq t-1$,

$$g(r + d/t, t) \geq \frac{(rt + d)(rt + d) - t(t - r)}{(rt + d)(2rt + 2d - 2t + r)}.$$

Combining Theorems 3 and 7 we have:

Corollary 4. If $\omega > 2$ is a rational number which is not an integer, then $g(\omega) = (\omega + 1)/(2\omega)$.

Theorem 8. For any given integers $\omega \geq 2$ and $t \geq \omega$,

$$g(\omega, t) \geq \frac{\omega t + t + 1}{\omega(2t + 1)} = 1 - \frac{(\omega - 1)(t + 1)}{\omega(2t + 1)}.$$

Combining Theorems 3 and 8 we have:

Corollary 5. For any given integer $\omega > 2$, $g(\omega) = (\omega + 1)/(2\omega)$.

All the results we obtained are for $t \geq \omega - 1$. The next theorem can be applied for $t < \omega - 1$.

Theorem 9. If c, ω, t are integers such that $1 \leq c \leq t-1$ and $2^{c-1}t - 2^{c-1}(c-2) + 1 \leq \omega \leq 2^c t - 2^c(c-1)$, then $g(\omega, t) \geq \frac{t-c+(t-1)\omega+1}{t-c+2(t-1)\omega+2}$.

IV. SERVERS WITH AT LEAST $t-1$ SINGLETONS

All the given lower bounds are improved with a construction which generalizes the one for $1 < \omega \leq 2$. The construction can be applied for all admissible pairs (ω, t) . For simplicity we will define and demonstrate it first for integer values of ω and later explain the modification needed for non-integer values of ω .

The construction uses ω ($\lceil \omega \rceil$ if not an integer) types of servers. Type T_r , $1 \leq r \leq \omega$, has $t-1$ singleton cells and one cell with a sum of $(r-1)t+1$ parts. For each type, all possible combinations of parts and sums are taken the same amount of times. Therefore, the number of servers in Type T_1 is a multiple of $\binom{\omega t}{t}$ and the number of servers in Type T_r , $2 \leq r \leq \omega$, is a multiple of $\binom{\omega t}{t-1} \binom{\omega t - t + 1}{(r-1)t+1}$. A part x_i is recovered from all the singleton cells, where it appears, and also by pairing servers as follows. We construct $s-1$ bipartite graphs, where bipartite graph r , G_r , $1 \leq r \leq \omega-1$, has two sides. The first side represents all the servers of Type T_r in which x_i is neither a singleton nor in a sum with other parts. The second side represents all the servers of Type T_{r+1} in which x_i participates in a sum with other parts. There is an edge between vertex v of the first side and vertex u of the second side if the $t-1$ singleton parts in v , and the $(r-1)t+1$ parts of the sum in the last cell of u are the rt parts in the sum of the last cell of v , excluding x_i . Each server of Type T_r will appear η_r times in the construction, for some integer η_r . The calculation of η_r should be performed in such a way that the bipartite graphs will all be all regular.

We start with a general solution for $\omega = 3$ to show that this method is much better than the previous ones. For $\omega = 3$ there are three types of servers T_1, T_2 , and T_3 .

In Type T₁, each server has t singletons. There are $\binom{3t-1}{t-1}$ combinations in which x_i is a singleton and $\binom{3t-1}{t}$ combinations in which x_i is not a singleton. Each combination will appear in $\binom{2t-1}{t-1}$ servers of Type T₁.

In Type T₂, each server has $t-1$ singletons and one cell with a sum of $t+1$ parts. There are $\binom{3t-1}{t-2}\binom{2t+1}{t}$ combinations in which x_i is a singleton, $\binom{3t-1}{t-1}\binom{2t}{t}$ combinations in which x_i is in a sum of $t+1$ parts, and $\binom{3t-1}{t-1}\binom{2t}{t-1}$ combinations in which x_i is neither a singleton nor in a sum of $t+1$ parts. Each combination will appear in exactly one server of Type T₂.

In Type T₃, each server has $t-1$ singletons and one cell with a sum of $2t+1$ parts. Hence, each part appears in each server either as a singleton or in a sum of $2t+1$ parts. There are $\binom{3t-1}{t-2}$ combinations in which x_i is a singleton, and $\binom{3t-1}{t-1}$ combinations in which x_i is in a sum of $2t+1$ parts. Each combination will appear in $8\binom{2t}{t-1}$ servers of Type T₃.

Now, we can form the two bipartite graphs and apply Corollary 2 to find the pairs from which x_i can be recovered. This implies that the rate of the code is $\frac{16t^2+7t+1}{24t^2+15t+3}$, which is much better than the rate of $\frac{4t+1}{6t+3}$ implied by Theorem 8. Hence, we have

Theorem 10.

$$g(3, t) \geq \frac{16t^2 + 7t + 1}{24t^2 + 15t + 3}.$$

The rate of the construction for each pair (ω, t) , which is a lower bound on $g(\omega, t)$, is given in the next theorem.

Theorem 11. *For any integers ω and t greater than one, the rate of the code by the construction is $\frac{\beta+\gamma}{\beta+2\gamma}$, where*

$$\beta = \prod_{\ell=1}^{\omega-1} (\ell t + 1) + (t-1) \sum_{r=2}^{\omega} \frac{(\omega-1)!}{(\omega-r)!} t^{r-2} \prod_{\ell=r}^{\omega-1} (\ell t + 1),$$

$$\gamma = \sum_{r=1}^{\omega-1} \frac{(\omega-1)!}{(\omega-1-r)!} t^{r-1} \prod_{\ell=r}^{\omega-1} (\ell t + 1).$$

Moreover, when $t \rightarrow \infty$ the rate meets the upper bound of Theorem 3, i.e. $(\omega+1)/(2\omega)$.

A careful analysis shows that the rate of this construction is larger from the rates of the previous constructions when $\omega > 2$ (see [2]).

If ω is not an integer, then the construction is very similar. We note that there is some flexibility in choosing the number of parts in each type (there was no such flexibility when ω was integer). In fact it seems that we have to use the same types of servers as in the case when ω is an integer, except for the last type. We can see this in an example with $t = 3$, $\omega = 7/3$, so $s = 7$. There are three types of servers T₁, T₂, and T₃.

In Type T₁, each server has 3 singletons. There are 15 combinations in which x_i is a singleton and 20 combinations in which x_i is not a singleton. Each combination will appear in three servers of Type T₁.

In Type T₂, each server has 2 singletons and one cell with a sum of four parts. There are 30 combinations in which x_i is a singleton, 60 combinations in which x_i is in a sum of four parts, and 15 combinations in which x_i is neither a singleton nor in a sum of four parts. Each combination will appear in exactly one server of Type T₂.

In Type T₃, each server has 3 singletons and one cell with a sum of seven parts. Hence, each part appears in each server either as a singleton or in a sum of seven parts. There are 6 combinations in which x_i is a singleton, and 15 combinations in which x_i is in a sum of seven parts. Each combination will appear in exactly one server of Type T₃.

Now, we can form the two bipartite graphs and apply Corollary 2 to find the pairs from which x_i can be recovered. This implies that the rate of the code is $\frac{52}{77}$ which is better than the $\frac{23}{35}$ rate implied by Theorem 7. The rates for other parameters are also better and a general rate for $w = 7/3$ is given by.

Theorem 12.

$$g(7/3, 3t) \geq \frac{160t^2 + 45t + 3}{224t^2 + 81t + 7}.$$

ACKNOWLEDGMENT

This work was supported in part by the EPSRC Grant EP/N022114/1.

REFERENCES

- [1] D. Augot, F. Levy-Dit-Vahel, and A. Shikfa, "A storage-efficient and robust private information retrieval scheme allowing few servers," in *Cryptology and Network Security*, pp. 222–239, Springer 2014.
- [2] S. R. Blackburn and T. Etzion, "PIR array codes with optimal PIR rate," *arxiv.org/abs/1607.00235*, July 2016.
- [3] T. H. Chan, S. Ho, and H. Yamamoto, "Private information retrieval for coded storage," *arxiv.org/abs/1410.5489*, October 2014.
- [4] B. Chor, O. Goldreich, E. Kushilevitz, and M. Sudan, "Private information retrieval," *Journal ACM*, pp. 965–981, 1998.
- [5] G. Fanti and K. Ramchandran, "Efficient private information retrieval over unsynchronized databases," *IEEE J. on Selected Topics in Signal Processing*, vol. 9, pp. 1229–1239, 2015.
- [6] A. Fazeli, A. Vardy, and E. Yaakobi, "Coded for distributed PIR with low storage overhead," *Proc. IEEE International Symposium on Information Theory*, pp. 2852–2856, Hong Kong, June 2015.
- [7] A. Fazeli, A. Vardy, and E. Yaakobi, "Private information retrieval without storage overhead: coding instead of replication," *arxiv.org/abs/1505.0624*, May 2015.
- [8] P. Hall, "On representatives of subsets," *Journal of London Mathematical Society*, vol. 10, pp. 26–30, 1935.
- [9] E. Kushilevitz and R. Ostrovsky, "Replication is not needed: Single database, computationally-private information retrieval," *Proc. 38-th IEEE Symp. Foundations Computer Science (FOCS)*, pp. 364–373, 1997.
- [10] N. Shah, K. Rashmi, and K. Ramchandran, "One extra bit of download ensures perfectly private information retrieval," *Proc. IEEE International Symposium on Information Theory*, pp. 856–860, Honolulu, Hawaii, June 2014.
- [11] S. Rao and A. Vardy, "Lower bound on the redundancy of PIR codes," *arxiv.org/abs/1605.01869*, May 2016.
- [12] R. Tajeddine and S. El Rouayheb, "Private information retrieval from MDS coded data in distributed storage systems," *arxiv.org/abs/1602.01458*, February 2016.
- [13] R. Tajeddine and S. El Rouayheb, "Private information retrieval from MDS coded data in distributed storage systems," *Proc. IEEE International Symposium on Information Theory*, pp. 1411–1415, Barcelona, Spain, 2016.